A naive question about quantum groups

Let G be a connected semisimple Lie group with finite center ; consider, using standard notation, its category $\mathcal{O} \subset \mathfrak{g}$-mod of BGG-modules, its category $\mathcal{H}$ of Harish-Chandra modules, its (complex) flag variety $G_{\mathbb{C}}/B$, its compact symmetric space G_c/K — and recall the following theorems.

(1) **Theorem** (BGG). For any simple finite dimensional object V of $\mathcal{O}$ there is a graded algebra isomorphism

$$\mathrm{Ext}^\bullet_{\mathcal{O}}(V,V) \simeq H^\bullet(G_{\mathbb{C}}/B, \mathbb{C}).$$

(2) **Theorem** (É. Cartan, Casselman). For any simple finite dimensional object V of $\mathcal{H}$ there is a graded algebra isomorphism

$$\mathrm{Ext}^\bullet_{\mathcal{H}}(V,V) \simeq H^\bullet(G_c/K, \mathbb{C}).$$

I think of these statements as being some kind of cohomological Schur Lemmas, whence the following definition.

(3) **Definition**. Let X be a topological space and $\mathcal{A}$ be a $\mathbb{C}$-category [see Bass [1] p. 57] equipped with a functor $F : \mathcal{A} \to \mathbb{C}$-mod. Then $\mathcal{A}$ is a **Schur category** over X if

$$\left.\begin{array}{c} V \in \mathcal{A} \\[1em] V \text{ simple} \\[1em] \dim FV < \infty \end{array}\right\} \Longrightarrow \mathrm{Ext}^\bullet(V,V) \simeq H^\bullet(X, \mathbb{C})$$

[isomorphism of graded algebras].

In this terminology Theorems (1) and (2) take the respective forms "$\mathcal{O}$ is a Schur category over $G_{\mathbb{C}}/B$" and "$\mathcal{H}$ is a Schur category over G_c/K".

The purpose of these few lines is to present a conjectural quantum analog of Theorem (1). To this end I proceed in two steps. First I define a category, denoted $\mathcal{O}(\mathfrak{g}, h, f)$, which is supposed to be a quantum analog of the category $\mathcal{O}$ [or more precisely of the category $\mathcal{O}$ "with weights in the root lattice"] ; then I conjecture that $\mathcal{O}(\mathfrak{g}, h, f)$ is a Schur category over the flag variety of $\mathfrak{g}$. The category $\mathcal{O}(\mathfrak{g}, h, f)$ will appear as a subcategory of a certain category $\mathcal{C}(\mathfrak{g}, h, f)$, which is itself a quantum analog of $(\mathfrak{g}, \mathfrak{h})$-mod [or more precisely of the category of $(\mathfrak{g}, \mathfrak{h})$-modules with weights in the root lattice]. Here are the details.

Let

$$\mathfrak{g} \text{ be a semismple Lie algebra,}$$

$$\alpha_1, ..., \alpha_r \text{ a basis of simple roots,}$$

$$(a_{ij}) \text{ the Cartan matrix } (i.e.\ a_{ij} = 2(\alpha_i|\alpha_j)/(\alpha_i|\alpha_i)),$$

$$h \text{ a complex number,}$$

$$f = (f_1, ..., f_r) \text{ a list of functions } f_i : \mathbb{Z}^r \to \mathbb{C}.$$

[It might help the reader to know before hand that the classical case will be obtained by putting $f_j(n) = \sum_i \ a_{ij} \ n_i$.]

Here starts the **definition of the category** $\mathcal{C}(\mathfrak{g}, h, f)$.

An object V of $\mathcal{C}(\mathfrak{g}, h, f)$ is a direct sum

$$V = \bigoplus_{n \in \mathbb{Z}^r} V(n)$$

of vector spaces equipped with endomorphisms x_i, y_i $(1 \leq i \leq r)$ satisfying

$$x_i V(n) \subset V(n + e_i),$$

$$y_i V(n) \subset V(n - e_i),$$

$$[x_i, y_j] v = \delta_{ij} \ f_j(n) v \quad \text{for} \quad v \in V(n),$$

where (e_i) is the canonical basis of $\mathbb{Z}^r$, and the **quantum Serre relations**, which putting

$$b(i,j) = 1 - a_{ij},$$

$$q(i) = \exp\left((\alpha_i | \alpha_i) \ \frac{h}{2} \right),$$

$$z_i = x_i \ \forall \ i \quad \text{or} \quad z_i = y_i \ \forall \ i,$$

take the form

$$\sum_{k=0}^{b(i,j)} (-1)^k \binom{b(i,j)}{k}_{q(i)} z_i^k \ z_j \ z_i^{b(i,j)-k} = 0 \quad \forall \ i \neq j.$$

[The classical case is of course given by $h = 0$.]

The morphisms are the obvious ones. [Here ends the definition of the category $\mathcal{C}(\mathfrak{g}, h, f)$.]

(4) **Definition of the category** $\mathcal{O}(\mathfrak{g}, h, f)$. Let $U_h(\mathfrak{n})$ be the algebra generated by the x_i subject to the quantum Serre relations. Then $\mathcal{O}(\mathfrak{g}, h, f)$ is the full subcategory of $\mathcal{C}(\mathfrak{g}, h, f)$ whose objects are $U_h(\mathfrak{n})$-finite and of finite length.

If $\mathcal{C}$ is $\mathbb{C}$-category and $\mathcal{B}$ a full sub-$\mathbb{C}$-category, say that $\mathcal{B}$ is **Ext-full** in $\mathcal{C}$ if for all $V, W \in \mathcal{B}$ the natural morphism

$$\mathrm{Ext}^\bullet_{\mathcal{B}}(V, W) \to \mathrm{Ext}^\bullet_{\mathcal{C}}(V, W)$$

is an isomorphism.

(5) **Conjectures**.

(a) The categories $\mathcal{O}(\mathfrak{g}, h, f)$ and $\mathcal{C}(\mathfrak{g}, h, f)$ are Schur categories [see (3)] over the flag variety of $\mathfrak{g}$,

(b) the inclusion $\mathcal{O}(\mathfrak{g}, h, f) \subset \mathcal{C}(\mathfrak{g}, h, f)$ is Ext-full.

In the classical case [*i.e.* $h = 0$, $f_j(n) = \sum_i a_{ij}\, n_i$] (a) is due to BGG [see Theorem (1)]. Fuser checked the conjecture for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. — Let $\mathcal{C}$ be either $\mathcal{C}(\mathfrak{g}, h, f)$ or $\mathcal{O}(\mathfrak{g}, h, f)$ and $\{V_i \mid i \in I\}$ a system of representatives of the simple objects in $\mathcal{C}$.

(6) **Conjecture**. The vector space $\oplus_{p,i,j} \operatorname{Ext}^p_{\mathcal{C}}(V_i, V_j)$ is a [nonunital] Koszul algebra.

This conjecture has been proved for $\mathfrak{sl}(2, \mathbb{C})$ by Fuser and for the classical category $\mathcal{O}$ by Beilinson, Ginzburg and Soergel (see [2]).

* * *

[1] Bass H., **Algebraic K-theory**, Benjamin, New York 1968.

[2] Beilinson A., Ginzburg V., Soergel W., Koszul duality patterns in representation theory, *J. Am. Math. Soc.* **9** No.2 (1996) 473-527.

This text and others are available at http://www.iecn.u-nancy.fr/~gaillard